Lecture 23 Weak Formulation of Some Equations

Weak Formulation of Elliptic Equations

- Laplace's Equation $-\Delta u = 0$ is the prototypical elliptic equation. This is also a very nicely behaved equation, so we begin here.

If $U \subset \mathbb{R}^n$ is a bdd domain and $u, \psi \in C_c^\infty(U)$,

$$\int_\Omega \psi \Delta u \, dx = -\int_\Omega \nabla u \cdot \nabla \psi \, dx$$ ~~(also works for $u \in H_0^1(U)$)~~

so the above "holds"

and

$$\langle u, \psi \rangle_{H^1} = \langle u, \psi \rangle_{L^2} + \int_\Omega \nabla u \cdot \nabla \psi \, dx$$

for $u \in H_0^1$, such that for the PDE

$$\begin{cases} -\Delta u = \lambda u + f \\ u|_{\partial U} = 0 \end{cases} \qquad f \in L^2, \; x \in U$$

we say u is a weak solution if

$$\int_\Omega [\nabla u \cdot \nabla \psi - \lambda u \psi - f \psi] \, dx = 0$$

for every $\psi \in C_c^\infty(U)$.

ex.) On $[0,2]$, Consider $\begin{aligned} u'' &= f \\ u(0) = u(2) &= 0 \end{aligned}$

with $f(x) = \begin{cases} x & 0 \le x \le 1 \\ -1 & 1 < x \le 2 \end{cases}$

We integrate on the piece-wise linear parts to guess

$$u(x) = \begin{cases} \frac{1}{6}x^3 - ax & 0 \le x \le 1 \\ -\frac{1}{2}x^2 + (a + 4/3)x - 2a - 2/3 & 1 < x \le 2 \end{cases}$$

as possible solutions by the boundary conditions.

We find a

$$\int_0^2 [u'\psi' + f\psi]dx = 0 \quad \text{for } \psi \in C_c^\infty$$

~~gettes~~ Integrating by parts

$$\int_0^2 u'\psi' dx = \int_0^1 (\tfrac{1}{2}x^2 - a)\psi'(x)dx + \int_1^2 (-x + a + 4/3)\psi'(x)dx$$

$$= (\tfrac{1}{2} - a)\psi(1) - \int_0^1 x\,\psi(x)dx - (a + 1/3)\psi(1)$$

$$+ \int_1^2 \psi(x)dx$$

$$= (\tfrac{1}{6} - 2a)\psi(1) - \int_0^2 f\psi dx$$

$$\Rightarrow \quad \tfrac{1}{6} - 2a = 0 \qquad \text{so} \qquad a = 1/12.$$

giving weak solution

$$u(x) = \begin{cases} \tfrac{1}{6}x^3 - \tfrac{1}{12}x & 0 \le x \le 1 \\ -\tfrac{1}{2}x^2 + \tfrac{17}{12}x - 5/6 & 1 < x \le 2. \end{cases}$$

→ We can show that this solution is unique (if we cover variational methods).

Weak Formulation of Evolution Equations

- The process here is very similar to the previous sections with some technicalities due to time derivatives.

- Consider first the wave eqn. on a bdd $U \subseteq \mathbb{R}^n$

$$\begin{cases} \partial^2 u/\partial t^2 - \Delta u = 0 \\ u|_{\partial U} = 0 \end{cases}$$

$$u|_{t=0} = g$$

$$\partial_t u|_{t=0} = h$$

If $u \in C^2([0,\infty) \times U)$ is a classical solution and $\psi \in C_c^\infty([0,\infty) \times U)$

$$\int_0^\infty \int_U [\psi][\partial^2 u/\partial t^2 - \Delta u]\,dx\,dt = 0$$

and $\int_U \psi(-\Delta u)\,dx = \int_U \nabla u \cdot \nabla \psi\,dx$

$$\int_0^\infty \psi\, \partial^2 u/\partial t^2\,dt =$$

$$= -h\psi|_{t=0} - \int_0^\infty \frac{\partial \psi}{\partial t}\,\partial u/\partial t\,dt$$

$$= -h\psi|_{t=0} + g\,\partial\psi/\partial t|_{t=0} + \int_0^\infty u\,\frac{\partial^2 \psi}{\partial t^2}\,dt$$

So

$$\int_0^\infty \int_U u\left[\frac{\partial^2 \psi}{\partial t^2}\right] + \nabla u \cdot \nabla \psi\,dx\,dt$$

$$= -\int_U g\,\frac{\partial \psi}{\partial t}\Big|_{t=0}\,dx + \int_U h\,\psi|_{t=0}\,dx$$

and this makes sense when $u(t,\cdot) \in H_0^1(U)$ for all t (Dirichlet Condition is imposed), $g \in L^1_{loc}(U)$, $h \in L^1_{loc}(U)$, and $\int_U \nabla u \cdot \nabla \psi\,dx$ is integrable in t (which follows if $\|u(t,\cdot)\|_{H^1}$ is integrable in t).

ex.)

Consider the 1-D wave equation with $h = 0$,

$$g(x) = \begin{cases} x & 0 \le x \le 1 \\ 2 - x & 1 \le x \le 2 \end{cases}$$ on $[0,2]$.

D'Alembert's Solution is given by extending g to an odd periodic function on $\mathbb{R}$ with period 4, and

$$u(t,x) = \tfrac{1}{2}[g(x+t) + g(x-t)]$$

As u is piecewise linear, $u(t,0) = u(t,2) = 0$, $u(t,\cdot) \in H_0^1((0,2))$ for all t.

- Then, the Solution above gives

$$\int_0^\infty \int_0^2 u\,\frac{\partial^2 \psi}{\partial t^2} + \frac{\partial u}{\partial x}\frac{\partial \psi}{\partial x}\,dx\,dt = \int_0^2 g\,\frac{\partial \psi}{\partial t}\,dt$$

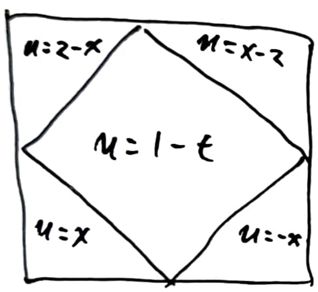
• To compute this, we must split the integral into pieces. For example, let us consider the case where φ is supported in $[0,1]\times(0,2)$

$$\int_0^1\int_0^2 u\,\frac{\partial^2\varphi}{\partial t^2}\,dx\,dt = \int_0^1\left[\int_0^{1-x} x\,\frac{\partial^2\varphi}{\partial t^2}\,dt + \int_{1-x}^{1}(1-t)\,\frac{\partial^2\varphi}{\partial t^2}\,dt\right]dx$$

$$+\int_1^2\left[\int_{1-x}^{1}(1-t)\,\frac{\partial^2\varphi}{\partial t^2}\,dt + \int_0^{x-1}(2-x)\,\frac{\partial^2\varphi}{\partial t^2}\,dt\right]dx$$

$$=\int_0^1\left[-x\,\frac{\partial\varphi}{\partial t}(0,x) - \varphi(1-x,x)\right]dx$$ (Some details involved)

$$+\int_1^2\left[-(2-x)\,\frac{\partial\varphi}{\partial t}(0,x) - \varphi(1-x,x)\right]dx$$



Similarly, $\int_0^1\int_0^2 \frac{\partial u}{\partial x}\cdot\frac{\partial\varphi}{\partial x}\,dx\,dt = \int_0^1 \varphi(1-x,x)\,dx + \int_1^2 \varphi(x-1,x)\,dx$

so $$\int_0^1\int_0^2\left[u\,\frac{\partial^2\varphi}{\partial t^2} + \frac{\partial u}{\partial x}\,\frac{\partial\varphi}{\partial x}\right]dx\,dt = \int_0^1 -x\,\frac{\partial\varphi}{\partial t}(0,x)\,dx + \int_1^2 -(2-x)\,\frac{\partial\varphi}{\partial t}(0,x)\,dx$$

$$= -\int_0^2 g(x)\,\frac{\partial\varphi}{\partial t}\,dx. \qquad \square$$

• In the case of Dirichlet conditions $g(x)=0$, $h(x)\neq 0$, we get slightly more to do.

• Let us consider the heat equation with Dirichlet conditions next

$$\frac{\partial u}{\partial t} - \Delta u = 0 \qquad u|_{x\in\partial U} = 0 \qquad u|_{t=0} = h$$

A similar process to the above gives

$$\int_0^\infty\int_U\left[-u\,\frac{\partial\varphi}{\partial t} + \nabla u\cdot\nabla\varphi\right]dx\,dt = \int_U h\,\varphi|_{t=0}\,dt$$

for all $\varphi\in C_c^\infty([0,\infty)\times U)$.

• Ex.) The Series Solution of discontinuous initial data is a weak Solution.

Consider the interval $U = (0, \pi)$, $h(x) \in L^2((0,\pi); \mathbb{R})$

with $h(x) = \sum a_k \phi_k(x)$ the Fourier Series.

On $(0,\pi)$, we recall $\phi_k(x) = \sqrt{2/\pi}\, \sin(kx)$ gave an orthonormal basis of L^2 for ~~~~ $k=1$ to ∞.

The heat Solution is $u(t,x) = \sum_{k=1}^{\infty} a_k e^{-k^2 t} \phi_k(x)$.

As we showed in Ch. 8, if h is Continuous, this is a classical Solution. (we showed C^1, actually)

For discontinuous h, we check for a weak Solution.
Pick $\psi \in C_c^\infty([0,\infty) \times (0,\pi); \mathbb{R})$.

Let $b_k(t) = \int_0^\pi \psi(t,x)\phi_k(x)\,dx$

So $\psi(t,x) = \sum_{k=1}^{\infty} b_k(t)\, \phi_k(x)$

with uniform and L^2 Convergence (b/c ψ is smooth)

Similarly, $\frac{\partial \psi}{\partial t} = \sum_{k=1}^{\infty} b_k'(t)\phi_k(x)$ is convergent uniformly.

By Parseval's Identity

$$\langle u, \tfrac{\partial \psi}{\partial t} \rangle = \int_0^\pi u \frac{\partial \psi}{\partial t}\, dx = \sum_{k=1}^{\infty} a_k e^{-k^2 t} b_k(t) \quad \text{for } t \geq 0.$$

and Similarly Setting up

$$\frac{\partial \psi}{\partial x}(t,x) = \sum_{k=1}^{\infty} b_k(t) \sqrt{\tfrac{2}{\pi}}\, \cos(kx)\,(k)$$

$$\frac{\partial u}{\partial x}(t,x) = \sum_{k=1}^{\infty} a_k(t) e^{-k^2 t} \sqrt{\tfrac{2}{\pi}}\, \cos(kx)\,(k)$$

Showing $\int_0^\pi \frac{\partial u}{\partial x} \frac{\partial \psi}{\partial x}\, dx = \sum_{k=1}^{\infty} k^2 a_k e^{-k^2 t} b_k(t)$

or that

$$\int_0^\infty \int_0^\pi \left[-u\frac{\partial \varphi}{\partial t} + \frac{\partial u}{\partial x}\frac{\partial \varphi}{\partial x}\right] dx\,dt =$$

$$\int_0^\infty \left[\sum_{k=1}^\infty a_k e^{-k^2 t}(k^2 b_k - b_k')\right] dt$$

- There is some nuance to be allowed here: we wish to swap the order of the sum and integral. I will not display the argument to show that we may, as it distracts slightly from the goal.

$$= \sum_{k=1}^\infty \int_0^\infty a_k e^{-k^2 t}(k^2 b_k - b_k')\,dt$$

$$= \sum_{k=1}^\infty \int_0^\infty a_k \frac{d}{dt}\left(e^{-k^2 t} b_k(t)\right) dt$$

$$= \sum_{k=1}^\infty a_k b_k(0)$$

Notice that this is $\int_0^\pi h(x)\varphi(0,x)dx = \sum_{k=1}^\infty a_k b_k(0)$ by Parseval's Id. again.